A mathematical proof for a ground-state identification criterion

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We give a mathematical proof for an identification criterion by a probability measure for the ground state among an infinite number of available states, or a finitely truncated number with appropriate boundary conditions, in a quantum adiabatic algorithm for Hilbert's tenth problem.

Background

In a quantum adiabatic algorithm for Hilbert's tenth problem [1], we have provided a mathematical proof in two dimensions for an identification criterion for the ground state of the final Hamiltonian. The criterion states that, in the case of no degeneracy for the final Hamiltonian H_P, the Fock state that has a measurement probability of greater than one-half *is* the ground state of H_P. At first, we thought that the proof was also valid for higher dimensions. Thanks to Warren Smith, who provided a counterexample in five dimensions [2], we now know that our proof is insufficient for higher dimensions. However, we have previously pointed out in our reply (in an earlier version of [3]): (i) that such a counterexample by Smith could be realised only because of the artefact resulted from the truncation, and its associated boundary conditions, in the dimensions of a dimensionally infinite Hilbert space; (ii) that a suitable choice of some complex parameters (α's in (3) below) would restore the identification criterion even for finite dimensions; (iii) and that we suspected, nevertheless, that our criterion would still be valid in *infinite* dimensions. In this short note, we present a mathematical proof for this last claim, and also for the case of finitely truncated Hilbert spaces with appropriate boundary conditions.

Introduction

Our quantum adiabatic algorithm [1] for a Diophantine equation, with K variables,

$$D(x_1, \ldots, x_K) \stackrel{?}{=} 0, \tag{1}$$

initially starts with a coherent state,

$$|\psi(0)\rangle = \bigotimes_{i=1}^{K} \left\{ \mathrm{e}^{-|\alpha_i|^2/2} \sum_{n_i=0}^{\infty} \frac{\alpha_i^{n_i}}{\sqrt{n_i!}} |n_i\rangle \right\}, \tag{2}$$

which is the nondegenerate ground state of a Hamiltonian H_I, with complex numbers $\alpha_i \neq 0$,

$$H_I = \sum_{i=1}^{K} \left(a_i^\dagger - \alpha_i^* \right) (a_i - \alpha_i) \,. \tag{3}$$

This Hamiltonian is then linearly extrapolated in a time T via a time-dependent Hamiltonian $\mathcal{H}(t)$,

$$\mathcal{H}(t) = H_I + \frac{t}{T}(H_P - H_I), \tag{4}$$

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to a final Hamiltonian H_P,

$$H_P \;=\; \left(D(a_1^\dagger a_1, \ldots, a_K^\dagger a_K)\right)^2, \tag{5}$$

which encodes in its ground state the information about the existence of solution of the Diophantine equation (1) [5].

In order to have the measurement probability of *any* excited state of H_P less than one-half for *any* time interval T, we can derive from [1] as part of the sufficient conditions the requirement that

$$\langle e(t)|H_P - H_I|f(t)\rangle \;\neq\; 0, \tag{6}$$

for all $0 < t < T$, for *any* pairs of orthogonal instantaneous eigenstates $|e(t)\rangle$ and $|f(t)\rangle$ of $\mathcal{H}(t)$.

Note that from the orthonormal instantaneous eigenstates,

$$\|\,|f(t)\rangle\| = \|\,|e(t)\rangle\| = 1, \tag{7}$$
$$\langle e(t)|f(t)\rangle = 0, \tag{8}$$
$$\mathcal{H}(t)|e(t)\rangle = E_e(t)|e(t)\rangle, \text{ and } \mathcal{H}(t)|f(t)\rangle = E_f(t)|f(t)\rangle, \tag{9}$$

and from the explicit expression for $\mathcal{H}(t)$ in (4), we can easily see that a violation of the condition (6) implies

$$\langle e(t_0)|H_P - H_I|f(t_0)\rangle = 0 \;\Leftrightarrow\; \text{both } \langle e(t_0)|H_P|f(t_0)\rangle = 0, \tag{10}$$
$$\text{and } \langle e(t_0)|H_I|f(t_0)\rangle = 0, \tag{11}$$

at some time t_0 for at least one pair of instantaneous eigenstates.

For two dimensions, there are only two eigenstates at each t, and the last two conditions above are unsatisfiable if the two end-point Hamiltonians do not commute [1],

$$[H_I, H_P] \;\neq\; 0. \tag{12}$$

For higher number of dimensions, this noncommutativity is no longer sufficient and we will have to resort to the explicit forms of H_I and H_P.

A mathematical proof for infinite dimensions

We present here a proof by contradiction that the condition (6) always holds with infinite dimensions. That is, we assume that the conditions (10) and (11) are satisfied at some time t_0, then derive from them some contradictions in order to conclude that (6) must hold. We restrict for simplicity to the case of one variable, $K = 1$ in (1); and the proof should be generalisable to other finite values of K.

We first expand the instantaneous eigenstates at this time t_0 in terms of the Fock states

$$|f(t_0)\rangle = \sum_{n=0}^{\infty} f_n|n\rangle, \text{ and } |e(t_0)\rangle = \sum_{n=0}^{\infty} e_n|n\rangle. \tag{13}$$

The conditions (8), and (10) can then be represented, respectively, as

$$\sum_{n=0}^{\infty} e_n^* f_n \;=\; 0, \tag{14}$$
$$\sum_{n=0}^{\infty} \left(D(n)\right)^2 e_n^* f_n \;=\; 0. \tag{15}$$

We next make use of an important theorem [6], reproduced here word by word,

Theorem .1 (Bosanquet-Henstock) *The necessary and sufficient conditions that*

$$\gamma(\omega) = \sum_{k=1}^{\infty} g_k(\omega) c_k \quad (\omega > \omega_0)$$

should tend to a finite limit as $\omega \to \infty$ whenever $\sum_{k=1}^{\infty} c_k = s$ is convergent are that

$$\begin{aligned} &(A) \sum_{k=1}^{\infty} |g_k(\omega) - g_{k+1}(\omega)| \le M \ \ \textit{for every} \ \ \omega > \omega_0, \\ &(B) \lim_{\omega\to\infty} g_k(\omega) = \beta_k \ \ \textit{for every fixed} \ \ k. \end{aligned}$$

Moreover

$$(C) \lim_{\omega\to\infty} \gamma(\omega) = \beta_1 s + \sum_{k=1}^{\infty} (\beta_k - \beta_{k+1})(s_k - s),$$

where $s_k = \sum_{r=1}^{k} c_r$, and the existence of either side of (C) implies that of the other, provided that $\sum_{k=1}^{\infty} c_k$ converges to s.

To exploit the theorem, we make the following identifications

$$\begin{aligned} c_k &\rightsquigarrow e_n^* f_n, && (16) \\ g_k(\omega) &\rightsquigarrow (D(n))^2, \ \ \text{independent of } \omega, && (17) \end{aligned}$$

whereupon, $\gamma(\omega)$ is identified as the infinite sum in (15). We can now see that the simultaneous convergence of the lhs of (14) and (15) are *not* compatible because the condition (A) of the theorem,

$$(A) \rightsquigarrow \sum_{n=0}^{\infty} \left| (D(n))^2 - (D(n+1))^2 \right| \le M, \tag{18}$$

cannot be satisfied for any (Diophantine) polynomial $D(n)$, except for a constant polynomial or for a finite truncation of the Hilbert space, $(D(n))^2 = 0$ for n greater than some N_0, in which case the infinite sum in (A) turns into a finite sum and the condition (A) is then automatically satisfied with some finite and sufficiently large M. Such a truncation is behind the counterexample by Smith [2, 3].

The only possibility now left for the simultaneous convergence of both (14) and (15) to zero is

$$e_n^* f_n = 0, \ \ \text{for all } n \text{ greater than some } M_0. \tag{19}$$

We next show that this last condition, in turn, contradicts the normalisation (7) of the eigenstates.

That $|f(t_0)\rangle$ is an eigenstate of $\mathcal{H}(t_0)$ in (9) can be expressed explicitly as follows,

$$\begin{aligned} \left(1 - \frac{t_0}{T}\right) H_I |f(t_0)\rangle &= \left(E_f(t_0) - \frac{t_0}{T} H_P\right) |f(t_0)\rangle, \\ \left(1 - \frac{t_0}{T}\right) \left(a^\dagger a - \alpha a^\dagger - \alpha^* a + |\alpha|^2\right) |f(t_0)\rangle &= \left(E_f(t_0) - \frac{t_0}{T} H_P\right) |f(t_0)\rangle, \end{aligned} \tag{20}$$

where we have substituted the explicit expression (3) for H_I with $K = 1$. The rhs of (20) is *diagonal* in the Fock basis; on the other hand, the k-th component of the lhs puts some restriction on the values of f_{k-1} and f_{k+1} through the action of $a^\dagger$ and a [7], resulting in,

$$-\left(1 - \frac{t_0}{T}\right)\left(\alpha\sqrt{k}\, f_{k-1} + \alpha^*\sqrt{k+1}\, f_{k+1}\right) = \left(E_f(t_0) - \frac{t_0}{T} (D(k))^2 - \left(1 - \frac{t_0}{T}\right)\left(k + |\alpha|^2\right)\right) f_k, \tag{21}$$

for $k = 0, 1, \ldots$ It is worth stressing here that this constraint (21) is only applicable to infinite dimensions. With some simple modifications of the coefficients of f_k's on the lhs, it can

also be applicable to a finitely truncated number of dimensions, truncated to some $N_{\max}$, with appropriate boundary conditions, such as the (anti)-periodic conditions, $a^\dagger|N_{\max}\rangle = \mp c|0\rangle$ and $a|0\rangle = \mp c^*|N_{\max}\rangle$ (with c is some *non-zero* complex number). In particular, the constraint (21) is not available to the aruptly truncated condition, $a^\dagger|N_{\max}\rangle = 0 = a|0\rangle$, in the counterexample offered by Smith [2].

Now focusing on f_n, say, the condition (19) implies that:

1. Either $f_n = 0$ for n greater than some $N_0 > M_0$ (so that with no restriction on e_n for $n \geq N_0$ we still have $e_n^* f_n = 0$):

   Applying (21) with $k = N_0$, we immediately have $f_{N_0-1} = 0$, for $t_0 < T$. Then recursively applying (21) again but with $k = N_0 - 1$ and so on, we must have for $t_0 < T$ and *all* n,

   $$f_n = 0,$$

   contradicting the *non-zero* normalisation of $|f(t_0)\rangle$ in (7)!

2. Or more than two consecutive elements of f_n vanishing, for instance $(\ldots, f_{q-1}, 0, 0, f_{q+2}, \ldots)$ (and $e_{q-1} = 0 = e_{q+2}$, etc.):

   Likewise, applying (21) with $k = q$ to have $f_{q-1} = 0$, for $t_0 < T$, and with $k = q+1$ to have $f_{q+2} = 0$. Then inductively applying (21) for larger and smaller values of k, we must have for $t_0 < T$ and *all* n,

   $$f_n = 0,$$

   again contradicting the *non-zero* normalisation of $|e(t_0)\rangle$ in (7)!

3. Or f_n asymptotically vanishes for every other value of n, for example $(\ldots, f_{q-1}, 0, f_{q+1}, 0, f_{q+3}, \ldots)$ (of course then $e_n \to (\ldots, 0, e_q, 0, e_{q+2}, 0, e_{q+4}, \ldots)$):

   Applying (21) with $k = q+1$, we have $f_{q+1} = 0$ and we are then back to the case above of more than two consecutive elements of f_n vanishing, which is once again in contradiction to the *non-zero* normalisation of $|f(t_0)\rangle$ in (7).

   Alternative, we apply (21) with $k = q$ to obtain for asymptotically large k,

   $$k\,|f_{k-1}|^2 = (k+1)\,|f_{k+1}|^2 \quad \Rightarrow \quad |f_n|^2 \sim k\,|f_{k-1}|^2\, O\left(\frac{1}{n}\right), \quad \text{for} \;\; n > k \gg 1,$$

   contradicting the *convergence* of the infinite sum $\sum_{n=k+1}^{\infty} |f_n|^2$ [8] in the normalisation condition (7).

4. Or f_n asymptotically vanishes for only one component, $(\ldots, f_{q-2}, f_{q-1}, 0, f_{q+1}, f_{q+2}, \ldots)$, then because of the constraint (19), we must have $(\ldots, 0, 0, e_q, 0, 0, \ldots)$ and are back to one of the three cases above for e_n with its contradiction to the *non-zero* normalisation of $|e(t_0)\rangle$.

5. Likewise, for the case f_n has no vanishing component asymptotically, we can switch our attention to e_n, which must asymptotically have some zero components because of (19), to derive some contradiction as in the above.

All in all, the condition (6) can never be violated.

Similar to those derived above for $K = 1$, the contradictions for higher numbers of variables K should also follow essentially from some key geometrical patterns of higher dimensional lattices, of which each site is labelled by K integer-valued coordinates. Likewise, the contradictions can be manifest in general as the violations of the non-zero normalisation condition, or of the convergence of the series in the nomralisation, or both.

Summary

In [1], we have argued that the sufficient conditions in two dimensions for the measurement probability for any T is always less than one-half,

$$|\langle\psi(T)|n_e\rangle|^2 \;<\; 0.5,$$

where $|n_e\rangle$ is an eigenstate (Fock state) of, but not the ground state of, the final Hamiltonian H_P, are that:

- $|\psi(0)\rangle = |\alpha\rangle$, the coherent ground state of the initial Hamiltonian H_I;
- $|\langle\psi(0)|n_e\rangle|^2 < 0.5$;
- $\langle e(t)|H_P - H_I|f(t)\rangle \neq 0$, for any pair of orthonormal instantaneous eigenstates of $\mathcal{H}(t)$, and for all $0 < t < T$.

Moving on to an infinite number of dimensions, we can, by effectively reducing the problem to two dimensions through the inductive consideration of pairs, each in turn, of orthonormal instantaneous eigenstates, show that the conditions above are sufficient even in the case of maximally constructive interference among the various pairwise transitional amplitudes to any given state.

On the one hand, we have shown above that the last sufficient condition (6) cannot be violated in a dimensionally infinite Hilbert space or a finitely truncated space with appropriate boundary conditions, ensuring that the measurement probability at T for *any* excited state, which is not occupied at the initial time, cannot be more than one-half, for *any* T. On the other hand, we know from the quantum adiabatic theorem that provided the time interpolation in (4) is sufficiently slow, that is, T sufficiently large, the measurement probability of the final ground state can be made arbitrarily close to one, provided we start out in the initial ground state. Combining these two, we thus arrive at an identification criterion for the ground state of H_P by a probability measure: *The final Fock state that has a measurement probability of greater than one-half, which can always be obtainable at some sufficiently large and finite T,* is *the final ground state, assuming no degeneracy.*

This key result, nonconstructively proven, together with the quantum adiabatic theorem are the main reasons behind the quantum computability of some classical and recursive non-computable, namely, Hilbert's tenth problem [1].

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[5] For simplicity, we assume that the ground state here is nondegenerate; see [1] for the general situation.

[6] Theorem (**4.2**, I), pp. 65-66, in [4].

[7] With $a^\dagger a|k\rangle = k|k\rangle$, $a^\dagger|k\rangle = \sqrt{k+1}|k+1\rangle$, and $a|k\rangle = \sqrt{k}|k-1\rangle$.

[8] As $\sum_{n=k+1}^{\infty}|f_n|^2 \sim k\,|f_{k-1}|^2 \sum_{n=k+1}(1/n)$, and $\sum_{n=k+1}(1/n)$ does not converge!